Symbol	Meaning
i	Individual or case (Level-1)
j	Group or cluster (Level-2)
$\gamma$	Represents a fixed effect
$\gamma_{00}$	Grand mean
$\beta_{0j}$	Random intercept
$\beta_{1j}$	Random slope
$\epsilon_{ij}$	Within-group variability (Level 1)
$v_{0j}$	Between-group variability (Level 2)
$ au_{00}$	Res. var. of Level-2 random intercepts
$\sigma^2$	Res. var. of Level-1 random intercepts

Concept	Symbol	Level
Raw data	$x_{ij}$	Level 1 and 2
Grand mean centered	$\ddot{x}_{ij}$	Level 1 and $2$
Group mean centered	$\dot{x}_{ij}$	Level 1
Group means	$\left  \begin{array}{c} \bar{x}_{.j} \\ \bar{x'}_{.j} \end{array} \right $	Level 2
Centered group means	$\bar{x'}_{.j}$	Level 2

## IMPORTANT

Independent variables:  $x_1, x_2, x_3...$ Dependent variable: Y

Level 1: within-group variation, represented by iLevel 2: **between-group** variation, represented by j

Note that Level 1 variables can carry both Level 1 and 2 variation. In  $x_{ij}$ , the subscript 0 represents the intercept and the 1 represents the slope. E.g.,  $\gamma_{01}$  denotes the first predictor of the intercept and  $\gamma_{12}$  represents the second predictor of the slope.

The *fixed effects* of the model are constant for all cases in the population and do not carry i or j subscripts. Effects that vary across Level-2 units are denoted by  $\gamma$  and  $\epsilon$ .

The random effects of the model vary across Level-1 and Level-2 units. Effects that vary across Level-2 units are denoted by v and au.

## Intercept only Regression

$$Y_i = \beta_0 + \epsilon_i$$

Simple linear regression

 $Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$ 

Multiple linear regression

 $Y_i = \beta_0 + \beta_1 x'_i + \epsilon_i$ 

## Moderation

 $Y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2$ 

Moderation can be re-written as:

$$\mathbf{Y}_{\mathbf{i}} = \beta_0 + \beta_1 x_1 + (\beta_2 + \beta_3 x_1) x_2 + \epsilon_i$$

Fixed effects model

$$\mathbf{Y}_{\mathbf{i}} = \beta_1 x_{1i} + \beta_2 x_{2i} + \ldots + \beta_p x_{pi} + \epsilon_i$$

 $\epsilon_i \sim_{iid} N(0, \sigma^2)$ 

# Random effects ANOVA

Level-1:

$$Y_{ij} = \beta_{0j} + \epsilon_{ij}$$
$$\epsilon_i \sim N(0, \sigma^2)$$

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Level-2:

$$\beta_{0j} = \gamma_{00} + v_{0j}$$
  
 $v_{0j} \sim_{iid} N(0, \tau_{00})$ 

Reduced:

$$Y_{ij} = \gamma_{00} + \upsilon_{0j} + \epsilon_{ij}$$

## Random intercept regression model Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$
$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + \upsilon_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$\upsilon_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{ij} + v_{0j} + \epsilon_{ij}$$

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Means as outcomes

Level-1:

$$Y_{ij} = \beta_{0j} + \epsilon_{ij}$$
$$\epsilon_i \sim N(0, \sigma^2)$$

Level-2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + v_{0j}$$
$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01}w_j + v_{0j} + \epsilon_{ij}$$

Intercepts as outcomes Level-1:  $Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$  $\epsilon_i \sim N(0, \sigma^2)$ 

Level-2:

$$\beta_{0j} = \gamma_{00} + \gamma_{01} w_j + v_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$v_{0j} \sim_{iid} N(0, \tau_{00})$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01}w_j + \gamma_{10}x_{ij} + v_{0j} + \epsilon_{ij}$$

Slopes as outcomes Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

Level-2:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + \gamma_{01} w_j + \upsilon_{0j} \\ \beta_{1j} &= \gamma_{10} + \gamma_{11} w_j + \upsilon_{1j} \\ \left(\begin{array}{c} \upsilon_{0j} \\ \upsilon_{1j} \end{array}\right) \stackrel{iid}{\sim} \left(\begin{array}{c} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} \ \tau_{10} \\ \tau_{10} \ \tau_{11} \end{bmatrix} \right) \end{aligned}$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{01}w_j + \gamma_{10}x_{ij} + \gamma_{11}w_jx_{ij} + v_{0j} + v_{1j}x_{ij} + \epsilon_{ij}$$

Simple Intercepts and Simple Slopes

$$\hat{\mathbf{y}} = \gamma_{00} + \gamma_{01}w + \gamma_{10}x + \gamma_{11}wx$$

And can be re-written as:

$$\hat{y} = (\gamma_{00} + \gamma_{01}w) + (\gamma_{10} + \gamma_{11}w)x$$

**Random Intercepts and Random Slopes Model** Level-1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + \epsilon_{ij}$$

Level-2:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + \upsilon_{0j} \\ \beta_{1j} &= \gamma_{10} + \upsilon_{1j} \\ \begin{pmatrix} \upsilon_{0j} \\ \upsilon_{1j} \end{pmatrix} & \stackrel{iid}{\sim} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \tau_{00} \ \tau_{10} \\ \tau_{10} \ \tau_{11} \end{bmatrix} \right) \end{aligned}$$

Reduced:

$$Y_{ij} = \gamma_{00} + \gamma_{10}x_{ij} + v_{0j} + v_{1j}x_{ij} + \epsilon_{ij}$$

# Calculations

Correlation between random slopes and random intercepts:

$$r = \frac{\tau_{10}}{\sqrt{\tau_{00}}\sqrt{\tau_{11}}}$$

Intraclass correlation:

$$ICC = \frac{\tau_{00}}{\tau_{00} + \sigma^2}$$

#### Interpretation

Regression coefficient:  $\beta_i$  represents the predicted change in Y for a 1-unit increase in  $x_1$ , holding  $x_2$  constant.

Centered regression coefficient:  $\beta_i$  is the influence if  $x'_1$  on Y when is  $x'_2$  is zero or for an individual at the mean of  $x_2$ .

Regression coefficient when an interaction is present:  $\beta_i$  is the change in Y for a 1-unit increase in  $x_1$  when  $x_2$  is zero.

## Fixed effects

The *fixed effects* of the model are constant for all cases in the population and do not carry *i* or *j* subscripts. The Greek symbol gamma ( $\gamma$ ) denotes fixed effects. E.g.,  $\gamma_{00}$  has no subscript and does not vary.

 $Y_{ij} = \gamma_{00} + \gamma_{10} Female_{ij} + \gamma_{20} Minority_{ij} + \gamma_{30} Female_{ij} * Minority_{ij} + v_{0j} + \epsilon_{ij}$ 

The average language score for non-minority males is equal to  $\gamma_{00}$ . Female non-minorities scored  $\gamma_{10}$  points higher/lower on the language test compared to male non-minorities. Male minorities scored  $\gamma_{20}$  higher/lower on the language test compared to male non-minorities. Finally, female minorities scored  $\gamma_{30}$  points higher/lower compared to male minorities.

#### Covariance parameter estimates

 $\tau_{00}$  can be interpreted as between-group variance or the covariance within cluster j, e.g., how much do students in the same class vary, or what percentage of the total variance is attributable to the differences between classrooms?

## Confidence interval

$$\beta_1 = -0.11, 95\% CI = [-0.22, -0.01]$$

The unknown population effect of mother's IQ on the number of children born to her  $(\beta_1)$  is expected to lie within 95% of similarly constructed intervals as [-0.22, -0.01].

## Intraclass correlation

Within a cluster, how correlated are the Level-1 units? Of the total variation of your outcome, what percentage is between-cluster variation?

# $\ensuremath{p\mbox{-value}}$ and test-statistic

 $\beta_1 = -0.11, t(8) = -2.48, p = .038.$ 

There is a significant negative effect of mother's IQ on the number of children born to her. A 1-unit increase in mother's IQ is associated with a decrease of 1.1 children born to her.

Over repeated samples, the probability of observing a test statistic

of -2.48 or more extreme, give that the null hypothesis of no effect of mother's IQ on children born to her is true, is 3.8%.